Recent Results on LCI Minimal Exponent

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Plan for the talk

1. Bernstein-Sato polynomials and minimal exponents for hypersurfaces
2. Bernstein-Sato polynomials and V-filtrations for arbitrary codimension
3. Kashiwara-Lichtin's result in hypersurface case
4. Implications for the minimal exponent
5. Higher codimension minimal exponent (definition and properties)
6. Proof sketches
7. A class of examples

Conventions Everything is over $\mathbb{C}$. The "ambient" variety X is smooth and connected of dimension n. The ring of linear differential operators on X is denoted $\mathcal{D}_X$ (think of the Weyl algebra $\mathbb{C}\langle t, \partial_t \rangle$ with $[\partial_t, t] = 1$)
The subvariety $Z \subseteq X$ will be defined by $f_1, \ldots, f_r \in \mathcal{O}_X(X)$.

Bernstein-Sato Polynomial (codim. 1)

- Let $r = 1, f = f_1$. Consider the ring $\mathcal{O}_X[\frac{1}{f}, s]$ with s a new variable.
- f^s = formal symbol generating a rank one, free $\mathcal{O}_X[\frac{1}{f}, s]$-module

$$\mathcal{N}_f = \mathcal{O}_X[\frac{1}{f}, s]f^s.$$

- This has an interesting action of $\mathcal{D}_X$: the obvious $\mathcal{O}_X$-structure, with a vector field $\tau \in \mathcal{T}_X$ acting by

$$\tau(f^s) = \frac{\tau(f)s}{f} f^s.$$

- Motivated by power rule and chain rule of calculus.

Bernstein-Sato Polynomial (codim. 1)

- If $f = x_i$, then $\partial_{x_j}(x_i^s) = \begin{cases} s x_i^{s-1} & i = j \\ 0 & i \neq j \end{cases}$.
- Motivated by this, can ask: is it always possible to reduce power by differentiating such that the coefficient lies in $\mathbb{C}[s]$?
- The answer is yes: Bernstein and Kashiwara showed (by different methods) that there exists a non-zero polynomial $b(s) \in \mathbb{C}[s]$ and $P(s) \in \mathcal{D}_X[s]$ such that

$$P(s) f^{s+1} = b(s) f^s.$$

- The monic such polynomial of least degree is the *Bernstein-Sato polynomial*, denoted $b_f(s)$.

Bernstein-Sato Polynomial (codim. 1)

Some easy examples:

- $f = x_i$, then $b_f(s) = s + 1$.
- $f = x_i x_j$, then $b_f(s) = (s+1)^2$.
- $f = x_i^2 + x_j^3$, then $b_f(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})$. (try it!)
- $f = \det(x_{ij})$ on $\mathbb{A}^{n^2}$, then (Capelli's Identity)

$$b_f(s) = (s+1)(s+2)(s+3)\dots(s+n).$$

Observe: always divisible by $(s+1)$. To see why:

$$\text{Set } s = -1, \quad b_f(-1)f^{-1} = P(-1)f^0 \in \mathcal{O}_X(X) \implies V(f) = \emptyset.$$

Bernstein-Sato Polynomial (codim. 1)

We can make some other observations, which are hard to prove:

- (Kashiwara) $b_f(-\gamma) = 0$ implies $\gamma \in \mathbb{Q}_{>0}$.
- (Lichtin-Kollár) $\min\{\gamma \mid b_f(-\gamma) = 0\} = \mathrm{lct}(f)$.
- (Briançon-Maisonobe) $b_f(s) = s + 1$ iff f defines a smooth hypersurface

Minimal Exponent of hypersurfaces

- As $s+1$ always divides $b_f(s)$, can consider $\widetilde{b}_f(s) = \frac{b_f(s)}{(s+1)}$.
- M. Saito defines the *minimal exponent of f* as

$$\widetilde{\alpha}(f) = \min\{\gamma \mid \widetilde{b}_f(-\gamma) = 0\}, \text{ (which is } \infty \text{ iff } b_f(s) = s+1).$$

- Clearly, we have $\mathrm{lct}(f) = \min\{1, \widetilde{\alpha}(f)\}$. Large $\widetilde{\alpha}$ means "less singular".
- Saito showed that $\widetilde{b}_f(-\gamma) = 0$ implies $\gamma \in [\widetilde{\alpha}(f), n - \widetilde{\alpha}(f)]$, in particular, if it is finite, we have $\widetilde{\alpha}(f) \leq \frac{n}{2}$
- This is achieved for $f = x_1^2 + \cdots + x_n^2$ (try to show: $b_f(s) = (s+1)(s+\frac{n}{2})$)

Another viewpoint for B-S Polynomials

Return to $r \geq 1$ case. To talk about Bernstein-Sato polynomials in higher codimension, it is beneficial to rephrase slightly:

- We have a $\mathcal{D}_{X\times\mathbb{A}^r}$-module $\mathcal{B}_f = \bigoplus_{\alpha\in\mathbb{N}^r} \mathcal{O}_X \partial_t^\alpha \delta_f$, with coordinates $t_1, \dots, t_r$ on $\mathbb{A}^r$.
- Action is given by:

$$t_i h\delta_f = hf_i\delta_f, \text{ using also } [\partial_t^\alpha, t_i] = \alpha_i \partial_t^{\alpha - e_i},$$

$$\tau(h\partial_t^\alpha \delta_f) = \tau(h)\partial_t^\alpha\delta_f - \sum_{i=1}^r \tau(f_i)h\partial_t^{\alpha+e_i}\delta_f, \quad \tau \in \mathcal{T}_X.$$

$$\partial_{t_i}(h\partial_t^\alpha\delta_f) = h\partial_t^{\alpha+e_i}\delta_f.$$

V-filtration on $\mathcal{D}_{X\times\mathbb{A}^r}$

Define a $\mathbb{Z}$-indexed, decreasing filtration

$$V^{\bullet}\mathcal{D}_{X\times\mathbb{A}^r} = \{\sum P_{\beta,\gamma}t^{\beta}\partial_t^{\gamma} \mid P_{\beta,\gamma}\in\mathcal{D}_X,\ |\beta|\geq \bullet + |\gamma|\}.$$

- For example, $t_i \in V^1, \partial_{t_i} \in V^{-1}$.
- We have $V^k\mathcal{D}\cdot V^j\mathcal{D}\subseteq V^{k+j}\mathcal{D}$.

G-Filtration on $\mathcal{B}_f$

- Define $G^\bullet\mathcal{B}_f = V^\bullet\mathcal{D}_{X\times\mathbb{A}^r}\cdot\delta_f$.
- If we set $s_i = -\partial_{t_i}t_i$, then δ_f satisfies the same relations as the formal symbol $f_1^{s_1}\dots f_r^{s_r}$. Let $s = \sum_{i=1}^r s_i \in V^0\mathcal{D}$.
- For $r = 1$, we have $V^0\mathcal{D} = \mathcal{D}_X[s,t]$ and $V^1\mathcal{D} = t\mathcal{D}_X[s,t] = \mathcal{D}_X[s,t]t$
- For $r \geq 1$, we define the *Bernstein-Sato polynomial* of $f_1,\dots,f_r$ by

$$b_f(s) = \text{ minimal polynomial of the action of } s \text{ on } \mathrm{Gr}_G^0\mathcal{B}_f.$$

- It is not hard to check that, for $r = 1$, this gives the same polynomial defined above.

V-filtration on $\mathcal{B}_f$

- There is another decreasing $\mathbb{Z}$-indexed filtration on $\mathcal{B}_f$ which is compatible with $V^\bullet\mathcal{D}$, denoted $V^\bullet\mathcal{B}_f$.
- This filtration was constructed for $\mathcal{B}_f$ by Malgrange and for arbitrary regular holonomic $\mathcal{D}$-modules by Kashiwara.
- For $r = 1$, the associated graded pieces are related (under Riemann-Hilbert) to the nearby and vanishing cycles along f.
- M. Saito refined this filtration to a discrete, left continuous $\mathbb{Q}$-indexed filtration.
- The important property of this refined filtration is that $s + \lambda$ is nilpotent on $\mathrm{Gr}_V^\lambda(\mathcal{B}_f) = V^\lambda\mathcal{B}_f / V^{>\lambda}\mathcal{B}_f$.

Relating V-filtration and B-S polynomials

- It is a fact that the induced V-filtration on $\mathrm{Gr}_G^0 \mathcal{B}_f$ is a finite $\mathbb{Q}$-indexed filtration.
- It satisfies

$$\mathrm{Gr}_V^{\gamma} \mathrm{Gr}_G^0 \mathcal{B}_f \neq 0 \text{ iff } (s+\gamma) \mid b_f(s).$$

- For any $u \in \mathcal{B}_f$, we can define $b_u(s)$, the Bernstein-Sato polynomial of u, as the minimal polynomial of the s action on

$$(V^0\mathcal{D} \cdot u)/(V^1\mathcal{D} \cdot u).$$

- Sabbah showed that the $\mathbb{Q}$-indexed V-filtration satisfies

$$V^{\lambda}\mathcal{B}_f = \{u \mid b_u(-\gamma) = 0 \implies \gamma \geq \lambda\}.$$

B-S polynomials and $\widetilde{\alpha}(f)$

- When $r = 1$, we consider $\partial_t^k \delta_f$ for $k \geq 1$.
- It is not too hard to show that $b_{\partial_t^k \delta_f}(s)$ and $\widetilde{b}_f(s-k)$ differ by a factor of $(s+1)$.
- Hence, the Bernstein-Sato polynomial of $\partial_t^k \delta_f$ gives information about $\widetilde{\alpha}(f)$.

Kashiwara-Lichtin's Result

- Recall that Kashiwara showed $b_f(-\gamma) = 0 \implies \gamma \in \mathbb{Q}_{>0}$. Really, related roots of $b_f(s)$ to numerical data from a resolution.
- Lichtin improved this argument to bring in the numerical data of the relative canonical divisor.
- Specifically, if $\pi : Y \to X$ is a log resolution of $(X, V(f))$ with $\pi^*(f) = \sum_I a_i E_i$ and $K_{Y/X} = \sum_I k_i E_i$, the argument of Lichtin showed

$$b_f(-\gamma) = 0 \implies \gamma = \frac{k_i + 1 + \ell}{a_i} \text{ for some } i \in I,\ \ell \in \mathbb{Z}_{\geq 0}.$$

- (D.-Mustață) We have

$$b_{\partial_t^k \delta_f}(-\gamma) = 0 \implies \gamma \in \mathbb{Z}_{\geq 1} \text{ or } \gamma = \frac{k_i + 1 + \ell}{a_i} - k$$

.

Applications to $\widetilde{\alpha}(f)$

- Putting all of this together, we get two interesting results: first, we can relate the V-filtration and the minimal exponent:

$$\widetilde{\alpha}(f) \geq 1 + k \iff \partial_t^k \delta_f \in V^1 \mathcal{B}_f.$$

  Also, we get the following lower bound: $\widetilde{\alpha}(f) \geq \min_{i \in I_{\mathrm{exc}}} \frac{k_i+1}{a_i}$.
- This need not be an equality: Kollár points out that the right hand side depends on the resolution.
- Mustață-Popa conjecture that, for any log resolution, there is some $i \in I$ such that $\widetilde{\alpha}(f) = \frac{k_i+1}{a_i}$.

Higher Codimension Case

- When $Z = V(f_1, \ldots, f_r)$ is a complete intersection of codimension r, we have
$$(s + r) \mid b_f(s).$$
- So we can define $\widetilde{b}_f(s) = \frac{b_f(s)}{(s+r)}$ as before, and consider
$$\widetilde{\gamma}(Z) = \min\{\gamma \mid \widetilde{b}_f(-\gamma) = 0\}.$$
- It is still the case that $\mathrm{lct}(X, Z) = \min\{\gamma \mid b_f(-\gamma) = 0\}$, so we get
$$\mathrm{lct}(X, Z) = \min\{r, \widetilde{\gamma}(Z)\}.$$
- This leads to the question: does $\widetilde{\gamma}(Z)$ relate to the V-filtration in the same way as in the $r = 1$ case?

Mustață's result and definition of $\widetilde{\alpha}(Z)$

- We have the following result of Mustață (even non-LCI): let $Y = X \times \mathbb{A}^r$ and let $g = \sum_{i=1}^r y_i f_i \in \mathcal{O}_Y(Y)$. Then

$$b_f(s) = \widetilde{b}_g(s).$$

- So g carries some information about the singularities of Z. We use this in the LCI case to define

$$\widetilde{\alpha}(Z) = \widetilde{\alpha}(g|_U), \quad U = X \times (\mathbb{A}^r \setminus \{0\}).$$

- Why? Because it works, and because

$$\mathrm{Sing}(g) = (Z \times \{0\}) \cup \Sigma, \quad \Sigma \text{ lies over } Z_{\mathrm{sing}}.$$

What do I mean by "it works"?

- With this definition, we have the following (something stronger, too, which we see below):

$$\widetilde{\alpha}(Z) \geq r + k \iff \partial_t^\beta \delta_f \in V^r \mathcal{B}_f \text{ for all } |\beta| \leq k.$$

- How could one prove this?
- Strengthening Mustață's result to relate the V-filtration on $\mathcal{B}_f$ and the "microlocal" V-filtration on $\mathcal{B}_g$.
- Use the known properties of V-filtrations for hypersurfaces and Kashiwara's equivalence (helps control $\mathcal{D}$-modules supported on $V(y_1, \dots, y_r)$).

Relating $\widetilde{\gamma}(Z)$ and $\widetilde{\alpha}(Z)$

- By definition, we always have $b_{g|_U}(s) \mid b_g(s)$, and so we get

$$\widetilde{b}_{g|_U}(s) \mid \widetilde{b}_g(s).$$

- Kashiwara's equivalence (and homogeneity of g) gives that

$$b_f(s) = \widetilde{b}_g(s) = \widetilde{b}_{g|_U}(s) \prod_J (s + r + j),$$

where $J \subseteq \mathbb{Z}_{\geq 0}$ is a finite subset.

- This gives $\widetilde{\alpha}(Z) \geq \widetilde{\gamma}(Z)$. One can argue that

$$\min\{\widetilde{\gamma}(Z), r+1\} = \min\{\widetilde{\alpha}(Z), r+1\},$$

and so the only problems arise when $\widetilde{\gamma}(Z) = r + j$ with $j \in \mathbb{Z}_{\geq 1}$.

Proving $\widetilde{\alpha}(Z) = \widetilde{\gamma}(Z)$

- We can assume $\widetilde{\alpha}(Z) > \widetilde{\gamma}(Z) = r + j$ for some $j \in \mathbb{Z}_{\geq 1}$.
- This is equivalent to $\partial_t^\beta \delta_f \in V^{>(r-1)}\mathcal{B}_f$ for all $|\beta| \leq j+1$.
- If we consider $\mathcal{K} = \bigcap_{i=1}^r \ker(\partial_{t_i} : \mathrm{Gr}_V^r \mathcal{B}_f \to \mathrm{Gr}_V^{r-1}\mathcal{B}_f)$, this says

$$\partial_t^\beta \delta_f \in \mathcal{K} \text{ for all } |\beta| \leq j.$$

- We want to show that $\mathrm{Gr}_V^{r+j}\mathrm{Gr}_G^0 \mathcal{B}_f = 0$, contradicting $(s+r+j) \mid b_f(s)$.

Finishing the proof

- The submodule $\mathcal{K} \subseteq \mathrm{Gr}_V^r \mathcal{B}_f$ has an induced G-filtration which (non-obviously) satisfies $G^0\mathcal{K} = \mathcal{K}$.
- In particular, $\mathrm{Gr}_G^{-j}\mathcal{K} = 0$, as $j \geq 1$.
- Hence, because $G^{-j}\mathcal{B}_f$ generated by $\partial_t^\beta \delta_f$ with $|\beta| \leq j$, we get

$$\mathrm{Gr}_V^r \mathrm{Gr}_G^{-j} \mathcal{B}_f = \mathrm{Gr}_G^{-j}\mathcal{K} = 0.$$

- Finally, using the nilpotency of $s + \chi$ on $\mathrm{Gr}_V^\chi \mathrm{Gr}_G^\ell \mathcal{B}_f$, we see that

$$\bigoplus_{|\beta|=j} \mathrm{Gr}_V^r \mathrm{Gr}_G^{-j} \mathcal{B}_f \xrightarrow{t^\beta} \mathrm{Gr}_V^{r+j} \mathrm{Gr}_G^0 \mathcal{B}_f \text{ is surjective,}$$

proving the claim.

A lower bound on $\widetilde{\alpha}(Z)$

We can provide a lower bound of $\widetilde{\alpha}(Z)$ if we take a stronger notion of resolution of the pair (X, Z).

These are the "strong factorizing resolutions" of Bravo-Villamayor (shown to exist in the generically reduced case).

If Z is generically reduced, this is a map $\pi : \widetilde{X} \to X$ such that

- π is proper and an isomorphism over $X \setminus Z_{\mathrm{sing}}$,
- $\widetilde{X}$ is smooth,
- The strict transform $\widetilde{Z}$ is smooth and has SNC with E, the exceptional,
- $\mathcal{I}_Z \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{I}_{\widetilde{Z}} \cdot \mathcal{O}_{\widetilde{X}}(-F)$ for some F supported on E.

A lower bound on $\widetilde{\alpha}(Z)$

- Chen-D.-Mustață showed the following: if $\pi : \widetilde{X} \to X$ is a strong factorizing resolution of (X, Z) and if $E = \sum_{j=1}^{N} E_j$, with $F = \sum_{j=1}^{N} a_j E_j$ and $K_{\widetilde{X}/X} = \sum_{j=1}^{N} k_j E_j$, we have

$$\widetilde{\alpha}(Z) \geq \min_j \frac{k_j + 1}{a_j}.$$

- The idea is to show that in the LCI case, a strong factorizing resolution for (X, Z) gives a log resolution of $(U, g|_U)$.

One class of examples

- We do not have many computations of $b_f(s)$ in the higher codimension case.
- In fact, even for $\widetilde{\alpha}(Z)$, initially we had only the following (which can be argued elementarily):
- Let $Z \subseteq \mathbb{A}^n$ be a complete intersection with isolated singularity at 0 defined by $f_1, \dots, f_r$ weighted homogeneous of (the same degree) d. Then

$$\widetilde{\alpha}(Z) = \frac{\sum_{i=1}^n w_i}{d}.$$

Broadening that class of examples

- Recently, Chen-D.-Mustață extended this to the following: if $Z = V(f_1, \dots, f_r) \subseteq \mathbb{A}^n$ complete intersection with isolated singularity at 0, with each f_i homogeneous of degree d_i, such that $d_1 \leq \cdots \leq d_r$ and so that if we set $H_i = V(f_i)$, then $H_i \setminus \{0\}$ is smooth and $\sum_{i=1}^r H_i$ is SNC, we have

$$\widetilde{\alpha}(Z) = p + \frac{1}{d_p}(n - d_1 - \cdots - d_p),$$

  where p is the minimal $i \leq r$ so that $d_1 + \cdots + d_i > n$.
- The difficult input is the lower bound, which comes from a strong factorizing resolution.
- We expect this to hold in the weighted homogeneous case, too, but cannot prove that at the moment.

Thank you

Thank you for your attention!